STABILIZED FLOW OF A VISCOUS LIQUID THROUGH A ROTATING CYLINDRICAL CHANNEL AT LOW ROSSBY NUMBER VALUES

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1. Formulation of the Problem. We will consider the problem of steady-state stabilized flow of an incompressible viscous liquid through a cylindrical channel with circular cross section of radius a, rotating at a constant angular velocity ω about an axis which intersects the channel axis and is perpendicular thereto.

We introduce the right-hand cartesian coordinate system $Ox^*y^*z^*$, rigidly fixed to the channel and oriented such that the axis Oz^* is directed along the channel axis in the direction of the flow, while the axis Oy^* coincides in direction with the angular velocity **ω**.

We will assume that liquid flow in the channel takes place under the action of a constant longitudinal gradient in the modified pressure $\partial\Pi/\partial z^* = \alpha$, and that the Rossby number Ro ($Ro = \omega_0^*/\omega a \ll 1$) is small.

With these assumptions liquid motion in the channel will be described by the following system of differential equations [1]:

$$\Delta\Delta\psi = 2R\partial w/\partial y,\ \Delta w = -2R\partial\psi/\partial y + 2, \tag{1.1}$$

where

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2;\ x = x^*/a;\ y = y^*/a;\ w = w^*/U; \tag{1.2}$$
$$\psi = \psi^*/Ua;\ R = \gamma^2 = \omega a^2/\nu;\ \Pi = p/\rho - (\omega^2/2)(x^{*2} + z^{*2});$$

$U = \alpha a^2/2\nu$ is the characteristic velocity; u^*, v^*, w^* are the projections of the relative velocity vector on the axes x^*, y^*, and z^*; ψ^* is the flow function of the secondary flow, related to u^* and v^* by the expressions $u^* = \partial\psi^*/\partial y^*$, $v^* = -\partial\psi^*/\partial x$; p is pressure; ρ is density; ν, kinematic viscosity. The boundary conditions for system (1.1) are the usual conditions of adhesion and impermeability:

$$w = \psi = \partial\psi/\partial x = \partial\psi/\partial y = 0 \text{ for } x^2 + y^2 = 1. \tag{1.3}$$

2. Integration of the Equations. If we take

$$\psi(x,\ y) = \chi(x,\ y) + y/R,\ w(x,\ y) = A_0 + f(x,\ y), \tag{2.1}$$

then, according to Eq. (1.1), χ and f must satisfy the following system of differential equations:

$$\Delta\Delta\chi = 2R\partial f/\partial y,\ \Delta f = -2R\partial\chi/\partial y. \tag{2.2}$$

Solutions of system (2.2) are provided by certain polynomials $G_n(x, y)$ and $T_n(x, y)$. A linear combination of these polynomials with multiplicative constants A_n will also be a solution of this system:

$$f_1 = \sum_{n=1}^{\infty} A_n G_n(x,\ y),\quad \chi_1 = -\sum_{n=1}^{\infty} A_n T_n(x,\ y)\frac{y}{R}, \tag{2.3}$$
$$G_1 = x^2,\quad G_2 = x^4,\quad G_3 = \frac{x^6}{6!} - \frac{1}{8}\frac{y^2}{R^2},$$
$$G_4 = \frac{x^8}{8!} - \frac{x^2y^2}{16R^2},\quad G_5 = \frac{x^{10}}{10!} - \frac{x^4y^2}{8\cdot 4!\,R^2} + \frac{3y^2}{32R^4},$$
$$G_6 = \frac{x^{12}}{12!} - \frac{x^6y^2}{8\cdot 6!\,R^2} + \frac{y^4 + 18x^2y^2}{2^4\cdot 4!\,R^4},\ \ldots$$

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$$T_1 = 1, \quad T_2 = 6x^2, \quad T_3 = \frac{x^4}{2\cdot 4!} - \frac{1}{8R^2},$$

$$T_4 = \frac{x^6}{2\cdot 6!} - \frac{3x^2 + y^2}{2\cdot 4!\,R^2}, \quad T_5 = \frac{x^8}{2\cdot 8!} - \frac{2x^2y^2 + x^4}{8\cdot 4!\,R^2} + \frac{3}{32R^4},$$

$$T_6 = \frac{x^{10}}{2\cdot 10!} - \frac{5x^4y^2 + x^6}{8\cdot 6!\,R^2} + \frac{9x^2 + 4y^2}{8\cdot 4!\,R^4}, \ldots$$

We will construct one more solution of system (2.2). In the plane of the channel cross section (the plane Oxy) we introduce a polar coordinate system r, φ, taking

$$x = r\cos\varphi, \; y = r\sin\varphi. \tag{2.4}$$

We will seek a solution of system (2.2) in polar coordinates in the form

$$f_2 = \sum_{n=0}^{\infty} g_{2n}(r)\cos 2n\varphi, \quad \chi_2 = \sum_{n=0}^{\infty} \tau_{2n+1}(r)\sin(2n+1)\varphi. \tag{2.5}$$

Substituting Eq. (2.5) in Eq. (2.2) and equating the corresponding expressions for sines and cosines with identical arguments, we get for definition of g_{2n} and τ_{2n+1} an infinite system of ordinary differential equations

$$\begin{cases} L_{2n}g_{2n} = R\left[\delta_n r^{2n-1}\dfrac{d}{dr}\left(\dfrac{\tau_{2n-1}}{r^{2n-1}}\right) - \dfrac{1}{r^{2n+1}}\dfrac{d}{dr}\left(r^{2n+1}\tau_{2n+1}\right)\right], \\ L_{2n+1}^2\tau_{2n+1} = R\left[\sigma_n r^{2n}\dfrac{d}{dr}\left(\dfrac{g_{2n}}{r^{2n}}\right) - \dfrac{1}{r^{2n+2}}\dfrac{d}{dr}\left(r^{2n+2}g_{2n+2}\right)\right], \end{cases} \tag{2.6}$$

$$\delta_n = \begin{cases} 0, & n = 0, \\ 1, & n \neq 0, \end{cases} \quad \sigma_n = \begin{cases} 2, & n = 0, \\ 1, & n \neq 1, \end{cases}$$

$$L_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \quad (n = 0, 1, 2, \ldots).$$

For arbitrary λ

$$\tau_{2n+1}(r) = B_{2n+1}J_{2n+1}(\lambda r), \; g_{2n}(r) = C_{2n}J_{2n}(\lambda r) \; (n = 0, 1, 2, \ldots), \tag{2.7}$$

where $J_m(\lambda r)$ is an m-th order Bessel function of the first kind, and B_{2n+1} and C_{2n} are coefficients interrelated as follows:

$$C_{2n} = (R/\lambda)(B_{2n+1} + \delta_n B_{2n-1}), \; B_{2n+1} = -(R/\lambda^3)(\sigma_n C_{2n} + C_{2n+2}), \tag{2.8}$$

is a solution of system (2.6). Using Eq. (2.8), all coefficients B_{2n+1} are sequentially expressed in terms of the coefficient B_1:

$$B_{2n+1} = F_{2n+1}(\eta)B_1, \; \eta = 2 - \lambda^4/R^2, \tag{2.9}$$

$$F_1 = 1, \; F_3 = -1 - \eta, \; F_5 = \eta^2 + \eta - 1, \; F_7 = 1 + 2\eta - \eta^2 - \eta^3,$$

$$F_9 = 1 - 2\eta - 3\eta^2 + \eta^3 + \eta^4,$$

$$F_{11} = -1 - 3\eta + 3\eta^2 + 4\eta^3 - \eta^4 - \eta^5, \ldots$$

Further, if we eliminate from Eq. (2.8) the coefficients C_{2n} and take $B_{2n+1}/B_{2n-1} = D_{2n+1}$, we obtain a system of recursive relationships relating to D_{2n+1}:

$$\begin{cases} D_3 = -(1 + \eta), \\ D_{2n+1} = \dfrac{-1}{\eta + D_{2n+3}} \end{cases} \quad (n = 1, 2, 3 \ldots); \tag{2.10}$$

D_3, on the one hand, is defined by the first expressions of Eq. (2.10), while according to the second relationship, it can be represented as a continuous fraction. Equating these expressions for D_3, we obtain

$$1 + \eta = \frac{1|}{|\eta} + \frac{-1|}{|\eta} + \frac{-1|}{|\eta} + \ldots, \tag{2.11}$$

the roots of which η_k (k = 1, 2, 3, ...) define possible values of λ_k:

$$\lambda_k = \pm\mu_k \exp(\pm 3\pi i/4), \quad \mu_k = \gamma\sqrt[4]{2 - \eta_k}, \quad i = \sqrt{-1}.$$

If the right side of Eq. (2.11) is replaced by a suitable fraction of order 1, we obtain an algebraic expression of order $l + 1$, all roots of which are real, different, and not larger than 2. To each such root η_k there corresponds a unique real value of μ_k and a pair of complex conjugate values λ_k. To each λ_k, in turn, there corresponds a definite $B_{1,k}$. We take

$$B_{1,k} = (M_k \pm iN_k)/2 \tag{2.12}$$

and will assume that $\lambda_k = \mu_k \exp(-3\pi i/4)$ corresponds to the plus sign, while $\lambda_k = \mu_k \exp(3\pi i/4)$ corresponds to the minus sign. (We will not consider the values $\lambda_k = -\mu_k \exp(\pm 3\pi i/4)$ since for these we have eigenfunctions differing only in sign.) If we now consider Eqs. (2.7)-(2.9) and the relationships between Bessel and Kelvin functions of the first sort

$$J_m(r\mu_k \exp(\pm 3\pi i/4)) = \mathrm{ber}_m(\mu_k r) \pm i\,\mathrm{bei}_m(\mu_k r),$$

we can show that the solution of Eq. (2.6) will be given by the following series:

$$\tau_{2n+1}(r) = \sum_{k=1}^{\infty} F_{2n+1}(\eta_k)\,[M_k\,\mathrm{ber}_{2n+1}(\mu_k r) + N_k\,\mathrm{bei}_{2n+1}(\mu_k r)],$$

$$g_{2n}(r) = \sum_{k=1}^{\infty} \frac{\mathrm{R}}{\sqrt{2}\mu_k}\,[\delta_n F_{2n-1}(\eta_k) + F_{2n+1}(\eta_k)]\,[(M_k - N_k)\,\mathrm{bei}_{2n}(\mu_k r) - (M_k + N_k)\,\mathrm{ber}_{2n}(\mu_k r)] \quad (n = 0, 1, 2\ldots).$$

In view of the linearity of system (2.2) the sum of its solutions (2.3) and (2.5) will also be a solution. From this it follows that if we consider Eq. (2.1) and transform to polar coordinates in Eq. (2.3), the solution of the original system (1.1) can be represented in the form

$$\psi = \frac{r\sin\varphi}{\mathrm{R}} + \sum_{n=0}^{\infty}\left[\tau_{2n+1}(r) - \frac{1}{\mathrm{R}}H_{2n+1}(r)\right]\sin(2n+1)\varphi; \tag{2.13}$$

$$w = A_0 + \sum_{n=0}^{\infty}[g_{2n}(r) + S_{2n}(r)]\cos 2n\varphi, \tag{2.14}$$

where

$$S_0 = A_1\frac{r^2}{2} + \frac{3}{8}A_2r^4 + A_3\left(\frac{5r^6}{2^4\cdot 6!} - \frac{r^2}{2^4\cdot\mathrm{R}^2}\right) + A_4\left(\frac{35r^8}{2^7\cdot 8!} - \frac{r^4}{2^7\mathrm{R}^2}\right) + A_5\left(\frac{63r^{10}}{2^8\cdot 10!} - \frac{r^6}{2^7\cdot 4!\,\mathrm{R}^2} + \frac{3r^2}{2^6\mathrm{R}^4}\right) + \ldots;$$

$$S_2 = A_1\frac{r^2}{2} + A_2\frac{r^4}{2} + A_3\left(\frac{r^6}{2^6\cdot 4!} + \frac{r^2}{2^4\mathrm{R}^2}\right) + A_4\frac{r^8}{2^7\cdot 6!} + A_5\left(\frac{105r^{10}}{2^8\cdot 10!} - \frac{r^6}{2^8\cdot 4!\mathrm{R}^2} - \frac{3r^2}{2^6\mathrm{R}^4}\right) + \ldots;$$

$$S_4 = A_2\frac{r^4}{8} + \frac{A_3r^6}{2^5\cdot 5!} + A_4\left(\frac{7r^8}{2^5\cdot 8!} + \frac{r^4}{2^7\mathrm{R}^2}\right) + A_5\left(\frac{15r^{10}}{2^6\cdot 10!} + \frac{r^6}{2^7\cdot 4!\,\mathrm{R}^2}\right) + \ldots;$$

$$S_6 = \frac{A_3r^6}{2^5\cdot 6!} + A_4\frac{r^8}{2^4\cdot 8!} + A_5\left(\frac{45r^{10}}{2^9\cdot 10!} + \frac{r^6}{2^8\cdot 4!\,\mathrm{R}^2}\right) + \ldots;$$

$$S_8 = A_4\frac{r^8}{2^7\cdot 8!} + A_5\frac{r^{10}}{2^9\cdot 9!} + \ldots;$$

$$S_{10} = A_5\frac{r^{10}}{2^9\cdot 10!} + \ldots;\ \ldots$$

$$H_1 = A_1r + \frac{3}{2}A_2r^3 + A_3\left(\frac{r^5}{2^4\cdot 4!} - \frac{r}{8\mathrm{R}^2}\right) + A_4\left(\frac{5r^7}{2^7\cdot 6!} - \frac{r^3}{2^5\mathrm{R}^2}\right) + A_5\left(\frac{7r^9}{2^8\cdot 8!} - \frac{r^5}{2^9\mathrm{R}^2} + \frac{3r}{2^5\mathrm{R}^4}\right) + \ldots;$$

$$H_3 = \frac{3}{2}A_2r^3 + A_3\frac{r^5}{2^8} + A_4\left(\frac{9r^9}{2^7\cdot 6!} - \frac{r^3}{4\cdot 4!\,\mathrm{R}^2}\right) + A_5\left(\frac{7r^9}{2^7\cdot 8!} - \frac{5r^5}{2^7\cdot 4!\,\mathrm{R}^2}\right) + \ldots;$$

$$H_5 = A_3\frac{r^5}{2^5\cdot 4!} + A_4\frac{5r^7}{2^7\cdot 6!} + A_5\left(\frac{5r^9}{2^7\cdot 8!} + \frac{r^5}{2^7\cdot 4!\,\mathrm{R}^2}\right) + \ldots;$$

$$H_7 = A_4\frac{r^7}{2^7\cdot 6!} + A_5\frac{7r^9}{2^9\cdot 8!} + \ldots;$$

$$H_9 = A_5 \frac{r^9}{2^9 \cdot 8!} + \ldots ; \ldots$$

The constants A_n, M_k, N_k, and A_0 appearing in Eqs. (2.13), (2.14) must be determined from boundary conditions (1.3). With consideration of Eq. (2.4), it is convenient to write the latter in the form

$$w = \psi = \partial\psi/\partial r = \partial\psi/\partial\varphi = 0 \quad \text{at} \quad r = 1. \tag{2.15}$$

Using Eqs. (2.13), (2.14) and satisfying the first three conditions of Eq. (2.15), we obtain an infinite system of algebraic equations for determination of the coefficients A_n, M_k, N_k

$$\begin{cases} S_{2n}(1) + g_{2n}(1) = 0, & \xi_n = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} \\ H_{2n+1}(1) - \mathrm{R}\tau_{2n+1}(1) = \xi_n, & \\ H'_{2n+1}(1) - \mathrm{R}\tau'_{2n+1}(1) = \xi_n & (n = 0, 1, 2, \ldots), \end{cases} \tag{2.16}$$

and also an equation for determination of A_0,

$$A_0 = -S_0(1) - g_0(1).$$

It can be shown that the fourth condition of Eq. (2.15) is then satisfied automatically. If roots of Eq. (2.11) are found and a solution of system (2.16) is obtained, then Eqs. (2.13), (2.14) are a solution of the problem formulated.

The question of convergence of the series of Eqs. (2.13), (2.14) remains open. We will only indicate that computer calculations performed for various R values of practical interest have shown that sufficient accuracy in calculation of $w(r, \varphi)$ and $\psi(r, \varphi)$ is realized even when only the first five coefficients A_{k-1}, M_k, and N_k are considered in Eqs. (2.13), (2.14). In calculating the values of the Kelvin function, depending on the value of the argument and index, either power series, Debye's formula, or Meissel's formula is used [2].

3. Channel Resistance. The mean flow velocity of the liquid through the rotating channel, referred to U, is defined by

$$w_0 = \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^1 w(r, \varphi)\, r dr.$$

Now substituting for w the value obtained from Eq. (2.14) and assuming that the corresponding series permit term-by-term integration, we obtain

$$w = A_0 - \sum_{k=1}^{\infty} \frac{2\mathrm{R}}{\mu_k^2} [M_k \operatorname{ber}_1(\mu_k) - N_k \operatorname{bei}_1(\mu_k)] + \Gamma,$$

$$\Gamma = \frac{A_1}{4} + \frac{A_2}{8} + A_3\left(\frac{5}{2^4 \cdot 6!} - \frac{1}{8\mathrm{R}^2}\right) + A_4\left(\frac{1}{2^{10} \cdot 6!} - \frac{1}{2^4 \cdot 4!\,\mathrm{R}^2}\right)$$

$$+ A_5\left(\frac{21}{2^{10} \cdot 10!} - \frac{1}{2^9 \cdot 4!\,\mathrm{R}^2} + \frac{1}{32\mathrm{R}^4}\right) + A_6\left(\frac{7}{2^7 \cdot 4!\,\mathrm{R}^4} - \frac{1}{2^{10} \cdot 6!\,\mathrm{R}^2} + \frac{1}{2^{12} \cdot 10!}\right) + \ldots$$

We define the hydraulic resistance coefficient of the rotating channel λ_w as

$$\lambda_\omega = 4a\alpha/(w_0^*)^2. \tag{3.1}$$

Now in Eq. (3.1) we replace α by the corresponding expression from Eq. (1.2) and form the ratio of the hydraulic resistance coefficients of the rotating channel λ_w and the nonmoving channel $\lambda_0 = 64/\mathrm{Re}$, and we will have

$$\lambda_\omega/\lambda_0 = 1/4w_0, \quad \mathrm{Re} = 2aw_0^*/\nu. \tag{3.2}$$

The solution obtained, when x^* is replaced by $x^* \pm b$ in Eq. (1.2), is also valid for a channel rotating with constant angular velocity ω relative to an axis which is perpendicular to the channel axis, but displaced therefrom by a distance b.

4. Results. The longitudinal distribution of the velocity w is symmetric relative to the axes Ox and Oy for all R values. Profiles of w/w_0 at the sections $\varphi = 0$ and $\varphi = \pi/2$ calculated for various R values (R = 1, 10, 25, 10^2, and 10^3) are shown in Figs. 1, 2, while

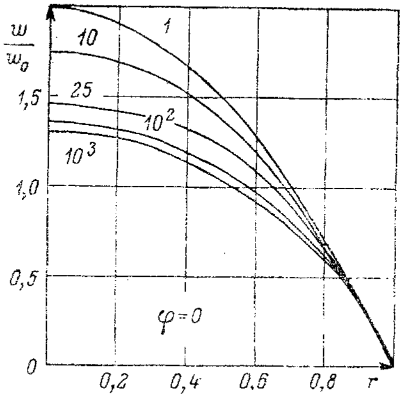

Fig. 1

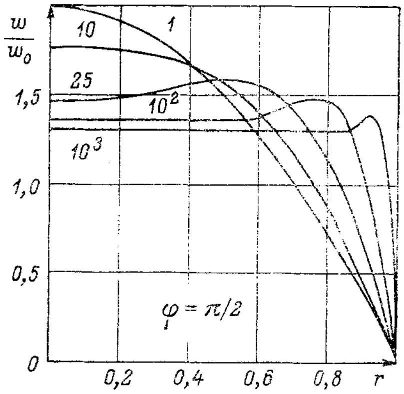

Fig. 2

similar profiles for $R = 10^3$ in the sections $\varphi = k\pi/12\ (k = 0, 1, 2, \ldots, 6)$ are shown in Fig. 3 (curves 0-6).

For $R \leqslant 1$ the w distribution is practically no different from the Poiseuille distribution for a fixed circular channel. For $R \gg 1$ in a circular channel, as contrasted to a rectangular one, the w distribution is quite nonuniform. At the same time in the diametral plane in the flow core w is described approximately by the expression

$$w/w_{\max} = (1 - x^2)^{3/4},$$

and w is practically constant everywhere in the plane $x = 0$, with the exception of a wall layer, where a distribution typical of Eckman's condition occurs [3].

A graph of the γ dependence of the longitudinal component of the velocity at the channel center $w(0, 0)/w_0$ is shown in Fig. 4.

The secondary flow in the cross section of the rotating channel (projections of liquid particle trajectories on the plane of the normal section) at all R values is a vortex pair, each half of which is a mirror reflection of the other half about the axis Ox. The center of each half of the vortex is on the axis Oy. For $R \ll 1$ the radius vector of the center of the vortices $r_0 = 0.447$, which coincides with the corresponding value for planar [4] and rectangular [5] channels. With increase in R the center of each vortex moves along the axis Oy toward the channel walls. The dependence of r_0 on γ is shown in Fig. 4.

The pattern of the secondary-flow flow lines, calculated with Eq. (2.13), coincides with the pattern obtained in [6] for $R \ll 1$, and is shown in Fig. 5 for $R = 10^3$ with a step $\Delta\psi/w_0 = -0.002$ for $y \geqslant 0$.

For $R \gg 1$ the secondary flow in the rotating-channel flow core is a uniform flow directed perpendicular to the axis of rotation. The value of the transverse velocity component in the channel center $u(0, 0)/w_0$ is shown as a function of γ in Fig. 4, where the γ dependence of the dimensionless flow rate Q circulating in each half of the channel cross section ($y \geqslant 0$ or $y \leqslant 0$) and referred to unit channel length is also shown:

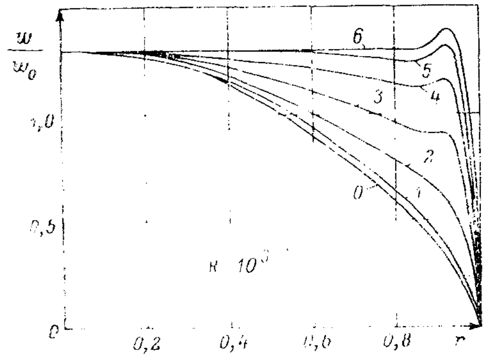

Fig. 3

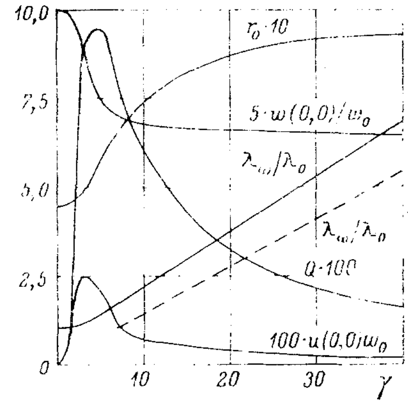

Fig. 4

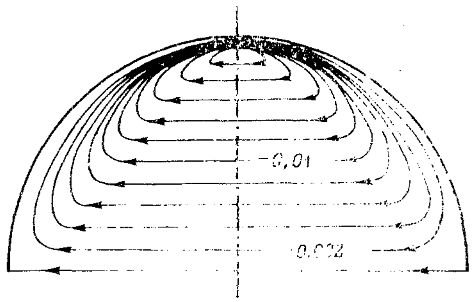

Fig. 5

$$Q = -\frac{1}{w_0}\int_0^{r_0} \psi_r'(r, \pi/2)\, dr = \frac{1}{w_0}\int_{r_0}^{1} \psi_r'(r, \pi/2)\, dr.$$

The maximum Q value $\simeq 0.095$ is attained at $\gamma \simeq 5$. From this is follows that for all γ values the amount of liquid circulating in half the rotating channel cross section does not exceed 10% of the flow in the main direction. From the graph of $\lambda_\omega/\lambda_0 = f(\gamma)$ shown in Fig. 4, according to calculations with Eq. (3.2), it follows that with increase in γ the hydraulic resistance of the rotating channel increases in comparison to a fixed channel, while at high γ the increase is practically linear. It is interesting that the functions $\lambda_\omega/\lambda_0 = f(\gamma)$ for channels with round and square [5, 7] cross sections are practically identical and agree well with the experiments of [8] for Ro < 1.

A comparison of calculations of the hydraulic resistance coefficient of the rotating channel with the formula

$$\lambda_\omega/\lambda_0 = 0.134\gamma/(1 - 1.05/\gamma), \tag{4.1}$$

proposed in [8] for $Ro \ll 1$ and $R \gg 1$ (dashed line of Fig. 4), with the results of the present analysis reveals that Eq. (4.1) gives lowered values of λ_ω.

5. Case of a Nonradial Channel. The solution obtained is easily generalized to the case of flow in a channel rotating with constant angular velocity Ω relative to an axis which intersects the channel axis, forming some angle therewith.

As in Sec. 1, we introduce a cartesian coordinate system $Ox^*y^*z^*$ rigidly attached to the channel and oriented such that its axis of rotation passes through the origin of the coordinate system and is located in the plane Oy^*z^*, while the axis Oz^* is directed along the channel axis in the direction of the flow. We denote the angle between the axis Oz^* and the vector $\mathbf{\Omega}$ by β. We then introduce the modified pressure with the expression

$$\Pi = p/\rho - (\Omega^2/2)[x^{*2} + (y^* \cos\beta - z^* \sin\beta)^2] + 2\Omega \cos\beta\psi^*$$

and take

$$\omega = \Omega \sin \beta. \qquad (5.1)$$

It can then be proved that with the assumptions made in Sec. 1, the flow of a viscous liquid through a rotating nonradial channel will again be described by Eq. (1.1) with boundary conditions (1.3). Thus it follows that the solution of the problem of the nonradial channel will again be defined by Eqs. (2.13), (2.14), in which ω is defined by Eq. (5.1).

LITERATURE CITED

1. H. P. Greenspan, Theory of Rotating Fluids, Cambridge Univ. Press (1968).
2. M. Abramovitz and I. Stegun, Handbook of Special Functions [Russian translation], Nauka, Moscow (1979).
3. G. S. Benton and D. Boyer, "Flow through a rapidly rotating conduit of arbitrary cross section," J. Fluid Mech., 26, Pt. 1 (1966).
4. O. N. Ovchinnikov and E. M. Smirnov, "Flow and heat exchange dynamics in a rotating slit-shaped channel," Inzh. Fiz. Zh., 35, No. 1 (1978).
5. O. N. Ovchinnikov, "Steady-state flow of a viscous liquid through a rotating radial channel at low Rossby number," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1980).
6. S. N. Barua, "Second flow in a rotating straight pipe," Proc. R. Soc., Ser. A, 227, 133 (1954).
7. O. N. Ovchinnikov, "Hydraulic resistance of a rotating channel," Izv. Akad. Nauk SSSR, Energ. Trans., No. 1 (1980).
8. H. Ito and K. Nanvu, "Flow in rotating channels with circular cross sections," Tr. Am. Ob-va Inzh. Mekh., Ser. D, Teor. Osnovy Inzh. Raschetov, 93, No. 3 (1971) [Russian translation of Trans. ASME].